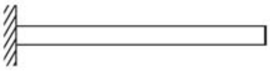
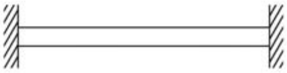
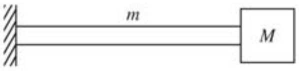
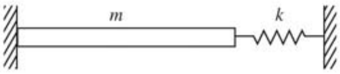
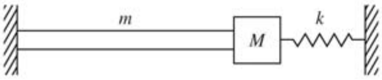
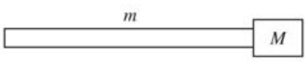
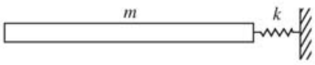
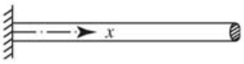
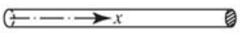
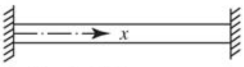
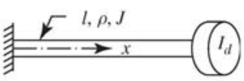
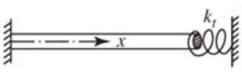
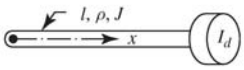
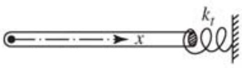
۱. جدول فرکانسهای طبیعی محوری میله‌ها

Table 9.1 Boundary Conditions of a Bar in Longitudinal Vibration

End conditions of the bar	Boundary conditions	Frequency equation	Mode shapes (normal functions)	Natural frequencies of vibration
1. Fixed–free 	$u(0,t)=0$ $\frac{\partial u}{\partial x}(l,t)=0$	$\cos\frac{\omega l}{c}=0$	$U_n(x)=$ $C_n\sin\frac{(2n+1)\pi x}{2l}$	$\omega_n=\frac{(2n+1)\pi c}{2l}$, $n=0,1,2,\ldots$
2. Fixed–fixed 	$u(0,t)=0$ $u(l,t)=0$	$\sin\frac{\omega l}{c}=0$	$U_n(x)=$ $C_n\sin\frac{n\pi x}{l}$	$\omega_n=\frac{n\pi c}{l}$, $n=1,2,3,\ldots$
3. Fixed–attached mass 	$u(0,t)=0$ $EA\frac{\partial u}{\partial x}(l,t)=$ $-M\frac{\partial^2 u}{\partial t^2}(l,t)$	$\alpha\tan\alpha=\beta$ $\alpha=\frac{\omega l}{c}$ $\beta=\frac{m}{M}$	$U_n(x)=$ $C_n\sin\frac{\omega_n x}{c}$	$\omega_n=\frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
4. Fixed–attached spring 	$u(0,t)=0$ $EA\frac{\partial u}{\partial x}(l,t)=$ $-ku(l,t)$	$\alpha\tan\alpha=-\gamma$ $\alpha=\frac{\omega l}{c}$ $\gamma=\frac{m\omega^2}{k}$	$U_n(x)=$ $C_n\sin\frac{\omega_n x}{c}$	$\omega_n=\frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
5. Fixed–attached spring and mass 	$u(0,t)=0$ $EA\frac{\partial u}{\partial x}(l,t)=$ $-M\frac{\partial^2 u}{\partial t^2}(l,t)$ $-ku(l,t)$	$\alpha\cot\alpha=$ $\frac{\alpha^2}{\beta}-\frac{k}{k_0}$ $\beta=\frac{m}{M}$ $k_0=\frac{AE}{l}$	$U_n(x)=$ $C_n\sin\frac{\omega_n x}{c}$	$\omega_n=\frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
6. Free–free 	$\frac{\partial u}{\partial x}(0,t)=0$ $\frac{\partial u}{\partial x}(l,t)=0$	$\sin\frac{\omega l}{c}=0$	$U_n(x)=$ $C_n\cos\frac{n\pi x}{l}$	$\omega_n=\frac{n\pi c}{l}$, $n=0,1,2,\ldots$
7. Free–attached mass 	$\frac{\partial u}{\partial x}(0,t)=0$ $EA\frac{\partial u}{\partial x}(l,t)=$ $-M\frac{\partial^2 u}{\partial t^2}(l,t)$	$\tan\alpha=-\alpha\beta$	$U_n(x)=$ $C_n\cos\frac{\omega_n x}{c}$	$\omega_n=\frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
8. Free–attached spring 	$\frac{\partial u}{\partial x}(0,t)=0$ $EA\frac{\partial u}{\partial x}(l,t)=$ $-ku(l,t)$	$\alpha\cot\alpha=\delta$ $\delta=\frac{AE}{lk}$	$U_n(x)=$ $C_n\cos\frac{\omega_n x}{c}$	$\omega_n=\frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$

۲. جدول فرکانسهای طبیعی پیچشی میله‌ها

Table 10.1 Boundary Conditions of a Uniform Shaft in Torsional Vibration

End conditions of shaft	Boundary conditions	Frequency equation	Mode shape (normal function)	Natural frequencies
1. Fixed–free 	$\theta(0,t) = 0$ $\frac{\partial\theta}{\partial x}(l,t) = 0$	$\cos\frac{\omega l}{c} = 0$	$\Theta_n(x) = C_n \sin\frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l}$, $n = 0, 1, 2, \ldots$
2. Free–free 	$\frac{\partial\theta}{\partial x}(0,t) = 0$ $\frac{\partial\theta}{\partial x}(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$\Theta_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$, $n = 0, 1, 2, \ldots$
3. Fixed–fixed 	$\theta(0,t) = 0$ $\theta(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$\Theta_n(x) = C_n \sin\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$, $n = 1, 2, 3, \ldots$
4. Fixed–disk 	$\theta(0,t) = 0$ $GJ\frac{\partial\theta}{\partial x}(l,t) = -I_d\frac{\partial^2\theta}{\partial t^2}(l,t)$	$\alpha\tan\alpha = \beta$ $\alpha = \frac{\omega l}{c}$ $\beta = \frac{\rho J l}{I_d}$	$\Theta_n(x) = C_n \sin\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n = 1, 2, 3, \ldots$
5. Fixed–torsional spring 	$\theta(0,t) = 0$ $GJ\frac{\partial\theta}{\partial x}(l,t) = -k_t\theta(l,t)$	$\alpha\tan\alpha = -\beta$ $\alpha = \frac{\omega l}{c}$ $\beta = \frac{\omega^2\rho J l}{k_t}$	$\Theta_n(x) = C_n \sin\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n = 1, 2, 3, \ldots$
6. Free–disk 	$\frac{\partial\theta}{\partial x}(0,t) = 0$ $GJ\frac{\partial\theta}{\partial x}(l,t) = -I_d\frac{\partial^2\theta}{\partial t^2}(l,t)$	$\alpha\cot\alpha = -\beta$ $\alpha = \frac{\omega l}{c}$ $\beta = \frac{\rho J l}{I_d}$	$\Theta_n(x) = C_n \cos\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n = 1, 2, 3, \ldots$
7. Free–torsional spring 	$\frac{\partial\theta}{\partial x}(0,t) = 0$ $GJ\frac{\partial\theta}{\partial x}(l,t) = -k_t\theta(l,t)$	$\alpha\cot\alpha = \beta$ $\alpha = \frac{\omega l}{c}$ $\beta = \frac{\omega^2\rho J l}{k_t}$	$\Theta_n(x) = C_n \cos\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n = 1, 2, 3, \ldots$
8. Disk–disk 	$GJ\frac{\partial\theta}{\partial x}(0,t) = I_1\frac{\partial^2\theta}{\partial t^2}(0,t)$ $GJ\frac{\partial\theta}{\partial x}(l,t) = -I_2\frac{\partial^2\theta}{\partial t^2}(l,t)$	$\tan\alpha = \frac{\alpha(\beta_1+\beta_2)}{(\alpha^2-\beta_1\beta_2)}$ $\alpha = \frac{\omega l}{c}$ $\beta_1 = \frac{I_0}{I_1} = \frac{\rho J l}{I_1}$ $\beta_2 = \frac{I_0}{I_2} = \frac{\rho J l}{I_2}$	$\Theta_n(x) = C_n\left(\cos\frac{\alpha_n x}{l} - \frac{\alpha_n}{\beta_1}\sin\frac{\alpha_n x}{l}\right)$	$\omega_n = \frac{\alpha_n c}{l}$, $n = 1, 2, 3, \ldots$

۳. مرجع

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